VC Dimension, VC Density, and the Sauer-Shelah Dichotomy

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References

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Set Systems

Definition

Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$. We call the pair $(X, \mathcal{S})$ a *set system*.

Definition

Given $A \subseteq X$, define

$$\mathcal{S} \cap A = \{B \cap A : B \in \mathcal{S}\}.$$

We say A is *shattered* by $\mathcal{S}$ iff: $\mathcal{S} \cap A = \mathcal{P}(A)$.

The Shatter Function and VC Dimension

Definition

The function $\pi_{\mathcal{S}} : \omega \to \omega$ given by

$$\pi_{\mathcal{S}}(n) = \max\{|\mathcal{S} \cap A| : A \in [X]^n\}$$

is called the *shatter function* of $\mathcal{S}$.

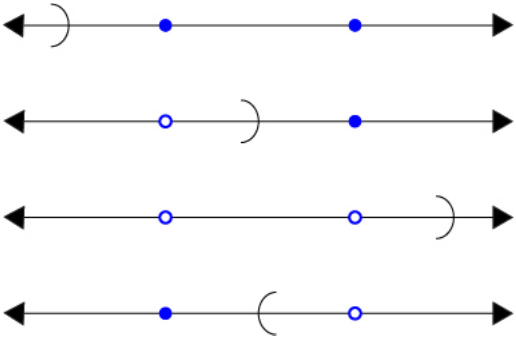
Definition

The *Vapnik-Chervonenkis (VC) dimension* of $\mathcal{S}$ is

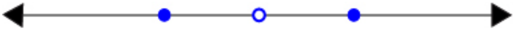
$$\begin{aligned} \mathrm{VC}(\mathcal{S}) &= \sup\{n < \omega : \mathcal{S} \text{ shatters some } A \in [X]^n\} \\ &= \sup\{n < \omega : \pi_{\mathcal{S}}(n) = 2^n\}. \end{aligned}$$

Example: $X = \mathbb{R}$, $\mathcal{S}$ = Half-Spaces

VC($\mathcal{S}$) ≥ 2:

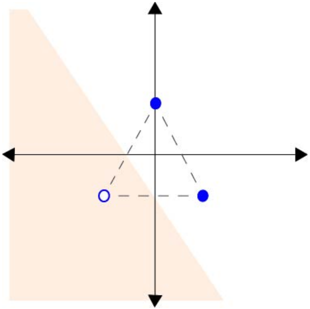


VC($\mathcal{S}$) < 3:



Example: $X = \mathbb{R}^2$, $\mathcal{S}$ = Half-Spaces

VC($\mathcal{S}$) ≥ 3 :



VC($\mathcal{S}$) < 4 :

The Sauer-Shelah Lemma

Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$.

Lemma (Sauer-Shelah)

If $\mathrm{VC}(\mathcal{S}) = d < \omega$*, then for all* $n \geq d$*, we have*

$$\pi_{\mathcal{S}}(n) \leq \binom{n}{0} + \cdots + \binom{n}{d} = O(n^d).$$

Proof: Suppose $\mathrm{VC}(\mathcal{S}) = d < \omega$, and fix $n > d$.

Let $A \in [X]^n$ such that $|\mathcal{S} \cap A| = \pi_{\mathcal{S}}(n)$, and let $(a_1, \ldots, a_n)$ enumerate A.

Inductively define $\mathcal{S}_0, \ldots, \mathcal{S}_n \subseteq \mathcal{P}(A)$ as follows:

Let $\mathcal{S}_0 = \mathcal{S} \cap A$.

To construct $\mathcal{S}_{i+1}$, remove a_{i+1} where possible; i.e,

$$\begin{aligned} \mathcal{S}_{i+1} = \ & \{B : B \in \mathcal{S}_i \text{ and } B \setminus \{a_{i+1}\} \in \mathcal{S}_i\} \\ & \cup \{B \setminus \{a_{i+1}\} : B \in \mathcal{S}_i \text{ and } B \setminus \{a_{i+1}\} \notin \mathcal{S}_i\}. \end{aligned}$$

Example: $\mathcal{S}_0 = \mathcal{S} \cap A$.

$A = \{$ a_1 a_2 a_3 a_4 a_5 $\}$

a_1	a_2	a_3	a_4	a_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
1	0	0	1	0
1	1	1	1	0

Example: Constructing $\mathcal{S}_1$.

	$A = \{$	a_1	a_2	a_3	a_4	a_5	$\}$
		0	0	0	0	0	
		0	0	0	0	1	
		0	0	0	1	0	
		0	0	1	0	0	
		0	0	1	0	1	
		0	0	1	1	0	
		0	0	1	1	1	
		0	1	0	0	0	
		0	1	0	1	0	
		0	1	1	0	0	
$\Rightarrow$		1	0	0	1	0	$\Leftarrow$
		1	1	1	1	0	

Example: Constructing $\mathcal{S}_1$..

$A = \{$	a_1	a_2	a_3	a_4	a_5	$\}$
	0	0	0	0	0	
	0	0	0	0	1	
$\rightarrow$	0	0	0	1	0	$\leftarrow$
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	1	1	1	
	0	1	0	0	0	
	0	1	0	1	0	
	0	1	1	0	0	
$\Rightarrow$	1	0	0	1	0	$\Leftarrow$
	1	1	1	1	0	

Example: Constructing $\mathcal{S}_1$...

$A = \{$	a_1	a_2	a_3	a_4	a_5	$\}$
	0	0	0	0	0	
	0	0	0	0	1	
$\rightarrow$	0	0	0	1	0	$\leftarrow$
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	1	1	1	
	0	1	0	0	0	
	0	1	0	1	0	
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		0	0	0	0	1	
		0	0	0	1	0	
		0	0	1	0	0	
		0	0	1	0	1	
		0	0	1	1	0	
		0	0	1	1	1	
		0	1	0	0	0	
		0	1	0	1	0	
		0	1	1	0	0	
		1	0	0	1	0	
$\Rightarrow$		1	1	1	1	0	$\Leftarrow$

Example: Constructing $\mathcal{S}_1$.....

$$A = \{ \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \}$$

	0	0	0	0	0	
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	1	1	1	
	0	1	0	0	0	
	0	1	0	1	0	
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	1	0	0	1	0	
$\Rightarrow$	0	1	1	1	0	$\Leftarrow$

Example: Constructing $\mathcal{S}_1$......

$A = \{$	a_1	a_2	a_3	a_4	a_5	$\}$
	0	0	0	0	0	
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	1	1	1	
	0	1	0	0	0	
	0	1	0	1	0	
	0	1	1	0	0	
	1	0	0	1	0	
	0	1	1	1	0	

Example: Constructing $\mathcal{S}_2$.

$$A = \{ \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \}$$

	a_1	a_2	a_3	a_4	a_5	
	0	0	0	0	0	
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	1	1	1	
$\Rightarrow$	0	1	0	0	0	$\Leftarrow$
	0	1	0	1	0	
	0	1	1	0	0	
	1	0	0	1	0	
	0	1	1	1	0	

Example: Constructing $\mathcal{S}_2$..

$A = \{$	a_1	a_2	a_3	a_4	a_5	$\}$
$\rightarrow$	0	0	0	0	0	$\leftarrow$
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	1	1	1	
$\Rightarrow$	0	1	0	0	0	$\Leftarrow$
	0	1	0	1	0	
	0	1	1	0	0	
	1	0	0	1	0	
	0	1	1	1	0	

Example: Constructing $\mathcal{S}_2$...

$A = \{$	a_1	a_2	a_3	a_4	a_5	$\}$
$\rightarrow$	0	0	0	0	0	$\leftarrow$
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	1	1	1	
$\Rightarrow$	0	1	0	0	0	$\Leftarrow$
	0	1	0	1	0	
	0	1	1	0	0	
	1	0	0	1	0	
	0	1	1	1	0	

Example: Constructing $\mathcal{S}_2$....

$$A = \{ \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \}$$

a_1	a_2	a_3	a_4	a_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
1	0	0	1	0
0	1	1	1	0

Example: Constructing $\mathcal{S}_3$.

	$A = \{$ a_1	a_2	a_3	a_4	a_5 $\}$	
	0	0	0	0	0	
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
$\Rightarrow$	0	0	1	1	1	$\Leftarrow$
	0	1	0	0	0	
	0	1	0	1	0	
	0	1	1	0	0	
	1	0	0	1	0	
	0	1	1	1	0	

Example: Constructing $\mathcal{S}_3$..

	$A = \{$ a_1	a_2	a_3	a_4	a_5 $\}$	
	0	0	0	0	0	
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
$\Rightarrow$	0	0	0	1	1	$\Leftarrow$
	0	1	0	0	0	
	0	1	0	1	0	
	0	1	1	0	0	
	1	0	0	1	0	
	0	1	1	1	0	

Example: Constructing $\mathcal{S}_3$...

$A = \{\ a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5\ \}$

a_1	a_2	a_3	a_4	a_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	0	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
1	0	0	1	0
0	1	1	1	0

Example: Constructing $\mathcal{S}_4$.

$A = \{ \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \}$

0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	0	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
1	0	0	0	0
0	1	1	1	0

Example: Constructing $\mathcal{S}_5$.

$A = \{$	a_1	a_1	a_3	a_4	a_5	$\}$
	0	0	0	0	0	
	0	0	0	0	1	
	0	0	0	1	0	
	0	0	1	0	0	
	0	0	1	0	1	
	0	0	1	1	0	
	0	0	0	1	1	
	0	1	0	0	0	
	0	1	0	1	0	
	0	1	1	0	0	
	1	0	0	0	0	
	0	1	1	1	0	

Back to proof...

Lemma (Sauer-Shelah)

If $\mathrm{VC}(\mathcal{S}) = d < \omega$, *then for all* $n \geq d$, *we have*

$$\pi_{\mathcal{S}}(n) \leq \binom{n}{0} + \cdots + \binom{n}{d} = O(n^d).$$

Proof (continued):

Notice that after stage $i+1$, we have the following:

- $|\mathcal{S}_{i+1}| = |\mathcal{S}_i|$.
- Given $A' \subseteq A$, if $\mathcal{S}_{i+1}$ shatters A', then S_i shatters A'.
- Given $B \in \mathcal{S}_{i+1}$ and $B' \subseteq B \cap \{a_1, \ldots, a_{i+1}\}$, $B' \cup (B \backslash \{a_1, \ldots, a_{i+1}\}) \in \mathcal{S}_{i+1}$.

Because of this, any member of $\mathcal{S}_n$ has cardinality at most d. $\square$

VC Density

Definition

The *VC density* of $\mathcal{S}$ is

$$\mathrm{vc}(\mathcal{S}) = \inf\left\{r \in \mathbb{R}^{>0} : \pi_{\mathcal{S}}(n) = O(n^r)\right\} = \limsup_{n\to\omega} \frac{\log \pi(n)}{\log n}.$$

Lemma (Sauer-Shelah)

If $\mathrm{VC}(\mathcal{S}) = d < \omega$, *then for all* $n \geq d$, *we have*

$$\pi_{\mathcal{S}}(n) \leq \binom{n}{0} + \cdots + \binom{n}{d} = O(n^d).$$

Corollary

$\mathrm{vc}(\mathcal{S}) \leq \mathrm{VC}(\mathcal{S})$.

Example: When $\mathcal{S}$ is “uniform,” VC dimension and VC density agree.

Let X be an infinite set and $\mathcal{S} = [X]^{\leq d}$ for some $d < \omega$.

We have

$$\pi_{\mathcal{S}}(n) = \binom{n}{0} + \cdots + \binom{n}{d},$$

so

$$\mathrm{VC}(\mathcal{S}) = \mathrm{vc}(\mathcal{S}) = d.$$

Example: VC dimension is more susceptible to local anomalies than VC density.

Let $X = \omega$, $m < \omega$, and $\mathcal{S} = \mathcal{P}(m)$.

It follows that

$$\pi_{\mathcal{S}}(n) = \begin{cases} 2^n & \text{if } n \leq m \\ 2^m & \text{otherwise.} \end{cases}$$

So

$$\mathrm{VC}(\mathcal{S}) = m$$

and

$$\mathrm{vc}(\mathcal{S}) = \limsup_{n \to \omega} \frac{\log 2^m}{\log n} = 0.$$

The Dual Shatter Function

Definition

Given $A_1, ..., A_n \subseteq X$, let $S(A_1, ..., A_n)$ denote the set of atoms in the Boolean algebra generated by $A_1, ..., A_n$. That is

$$S(A_1, \cdots, A_n) = \left\{ \bigcap_{i=1}^{n} A_i^{\sigma(i)} : \sigma \in {}^{n}2 \right\} \setminus \varnothing$$

where $A_i^1 = A_i$ and $A_i^0 = X \setminus A_i$.

Definition

The function $\pi^*_{\mathcal{S}} : \omega \to \omega$ given by

$$\pi^*_{\mathcal{S}}(n) = \max\{|S(A_1, ..., A_n)| : A_1, ...A_n \in \mathcal{S}\}$$

is called the *dual shatter function* of $\mathcal{S}$.

Dual VC Dimension and Dual VC Density

Definition

The *dual VC dimension* of $\mathcal{S}$ is

$$\mathrm{VC}^*(\mathcal{S}) = \sup\left\{n < \omega : \pi^*_{\mathcal{S}}(n) = 2^n\right\}.$$

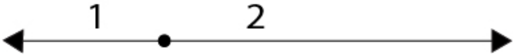
Definition

The *dual VC density* of $\mathcal{S}$ is

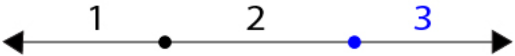
$$\mathrm{vc}^*(\mathcal{S}) = \inf\left\{r \in \mathbb{R}^{>0} : \pi^*_{\mathcal{S}}(n) = O(n^r)\right\}.$$

Example: $X = \mathbb{R}$, $\mathcal{S} =$ Half-Spaces

$\mathrm{VC}^*(\mathcal{S}) \geq 1$:

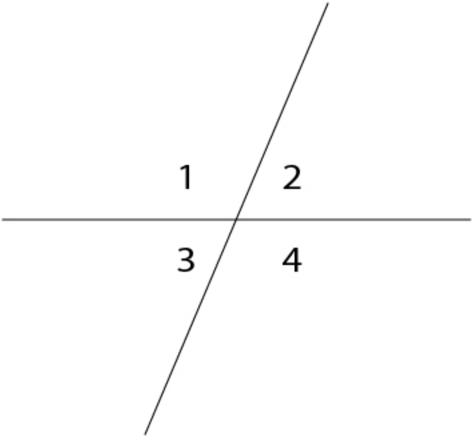


$\mathrm{VC}^*(\mathcal{S}) < 2$:



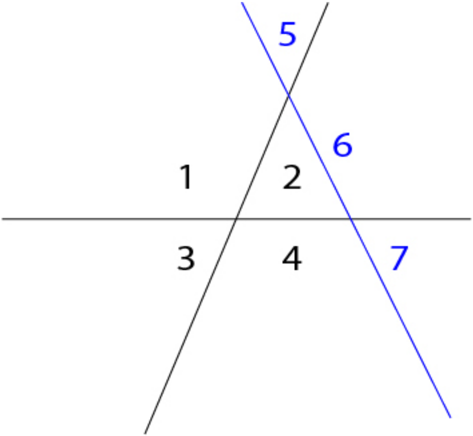
Example: $X = \mathbb{R}^2$, $\mathcal{S}$ = Half-Spaces

$\mathrm{VC}^*(\mathcal{S}) \geq 2$:



Example: $X = \mathbb{R}^2$, $\mathcal{S}$ = Half-Spaces

$\mathrm{VC}^*(\mathcal{S}) < 3$:



Set Systems in a Model-Theoretic Context

Consider a sorted language $\mathcal{L}$ with sorts indexed by I.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure with domains $(M_i : i \in I)$.

Definition

Given an $\mathcal{L}$-formula $\phi(x, y)$ where $x = (x_1^{i_1}, ..., x_s^{i_s})$ and $y = (y_1^{j_1}, ..., y_t^{j_t})$, define

$$\mathcal{S}_\phi = \{\phi(X, b) : b \in Y\}$$

where $X = M_{i_1} \times \cdots \times M_{i_s}$ and $Y = M_{j_1} \times \cdots \times M_{j_t}$.

It follows that $(X, \mathcal{S}_\phi)$ is a set system. To ease notation, we let:

π_ϕ denote $\pi_{\mathcal{S}_\phi}$, $VC(\phi)$ denote $VC(\mathcal{S}_\phi)$, and $vc(\phi)$ denote $vc(\mathcal{S}_\phi)$.

Similarly, we use π_ϕ^* for $\pi_{\mathcal{S}_\phi}^*$, $\mathrm{VC}^*(\phi)$ for $\mathrm{VC}^*(\mathcal{S}_\phi)$, and $\mathrm{vc}^*(\phi)$ for $\mathrm{vc}^*(\mathcal{S}_\phi)$.

The dual shatter function of ϕ is really counting ϕ-types.

By definition, we have $\pi^*_\phi(n) = \max\{|S(\phi(X,b) : b \in B)| : B \in [Y]^n\}$.

Let $B \in [Y]^n$. Recall that

$$S(\phi(X,b) : b \in B) = \left\{ \bigcap_{b \in B} \phi^{\sigma(b)}(X,b) : \sigma \in {}^{B}2 \right\} \setminus \varnothing.$$

There is a bijection

$$S(\phi(X,b) : b \in B) \longrightarrow \left\{ \mathrm{tp}_\phi(a/B) : a \in X \right\} = S_\phi(B)$$

given by

$$\bigcap_{b \in B} \phi^{\sigma(b)}(X,b) \longmapsto \left\{ \phi^{\sigma(b)}(x,b) : b \in B \right\}.$$

It follows that

$$|S(\phi(X,b) : b \in B)| = |S_\phi(B)|.$$

The Dual of a Formula

Definition

We call a formula $\phi(x; y)$ a *partitioned formula* with *object variable(s)* $x = (x_1, ..., x_s)$ and *parameter variable(s)* $y = (y_1, ..., y_t)$.

Definition

We let $\phi^*(y; x)$ denote the *dual* of $\phi(x; y)$, meaning $\phi^*(y; x)$ is $\phi(x; y)$ but we view y as the object and x as the parameter.

It follows that

$$\begin{aligned} \mathcal{S}_{\phi^*} &= \{\phi^*(Y, a) : a \in X\} \\ &= \{\phi(a, Y) : a \in X\}. \end{aligned}$$

The shatter function of ϕ^* is also counting ϕ-types.

By definition, we have $\pi_{\phi^*}(n) = \max \{|\mathcal{S}_{\phi^*} \cap B| : B \in [Y]^n\}$.

Let $B \in [Y]^n$. It follows that

$$\begin{aligned} \mathcal{S}_{\phi^*} \cap B &= \{\phi^*(B, a) : a \in X\} \\ &= \{\phi(a, B) : a \in X\} \end{aligned}$$

There is a bijection

$$\{\phi(a, B) : a \in X\} \longrightarrow \{\mathrm{tp}_\phi(a/B) : a \in X\} = S_\phi(B)$$

given by

$$\phi(a, B) \longmapsto \mathrm{tp}_\phi(a/B).$$

It follows that

$$|\mathcal{S}_{\phi^*} \cap B| = |S_\phi(B)|.$$

Duality in a Model-Theoretic Context

Lemma

The dual shatter function of ϕ is the shatter function of ϕ^*. That is $\pi^*_\phi = \pi_{\phi^*}$.

Proof: For all $n < \omega$, we have

$$\begin{aligned}
\pi^*_\phi(n) &= \max\{|S(\phi(X,b) : b \in B)| : B \in [Y]^n\} \\
&= \max\{|S_\phi(B)| : B \in [Y]^n\} \\
&= \max\{|\mathcal{S}_{\phi^*} \cap B| : B \in [Y]^n\} \\
&= \pi_{\phi^*}(n).
\end{aligned}$$

$\square$

Corollary

$\mathrm{VC}^*(\phi) = \mathrm{VC}(\phi^*)$ *and* $\mathrm{vc}^*(\phi) = \mathrm{vc}(\phi^*)$.

$\mathrm{VC}(\phi) < \omega \iff \mathrm{VC}^*(\phi) < \omega$

Lemma

$\mathrm{VC}(\phi) < 2^{\mathrm{VC}^*(\phi)+1}$.

Proof: Suppose $\mathrm{VC}(\phi) \geq 2^n$, there exists $A \in [X]^{2^n}$ shattered by $\mathcal{S}_\phi$.

Let $\{a_J : J \subseteq n\}$ enumerate A.

For all $i < n$, let $b_i \in Y$ such that $\mathcal{M} \models \phi(a_J, b_i) \iff i \in J$.

Let $B = \{b_i : i < n\}$.

It follows that $\mathcal{S}_{\phi^*}$ shatters B, so $\mathrm{VC}(\phi^*) \geq n$. □

Corollary

$\mathrm{VC}^*(\phi) < 2^{\mathrm{VC}(\phi)+1}$.

Corollary

$\mathrm{VC}(\phi) < \omega \iff \mathrm{VC}^*(\phi) < \omega$.

VC Dimension, VC Density, and the Sauer-Shelah Dichotomy

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The Dual Shatter Function

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$$S(A_1, \cdots, A_n) = \left\{ \bigcap_{i=1}^{n} A_i^{\sigma(i)} : \sigma \in {}^{n}2 \right\} \setminus \varnothing$$

where $A_i^1 = A_i$ and $A_i^0 = X \setminus A_i$.

Definition

The function $\pi^*_{\mathcal{S}} : \omega \to \omega$ given by

$$\pi^*_{\mathcal{S}}(n) = \max\{|S(A_1, ..., A_n)| : A_1, ...A_n \in \mathcal{S}\}$$

is called the *dual shatter function* of $\mathcal{S}$.

Dual VC Dimension and Dual VC Density

Definition

The *dual VC dimension* of $\mathcal{S}$ is

$$\mathrm{VC}^*(\mathcal{S}) = \sup\{n < \omega : \pi^*_{\mathcal{S}}(n) = 2^n\}.$$

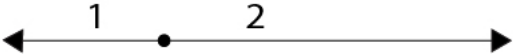
Definition

The *dual VC density* of $\mathcal{S}$ is

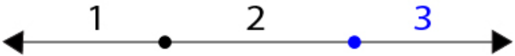
$$\mathrm{vc}^*(\mathcal{S}) = \inf\left\{r \in \mathbb{R}^{>0} : \pi^*_{\mathcal{S}}(n) = O(n^r)\right\}.$$

Example: $X = \mathbb{R}$, $\mathcal{S} =$ Half-Spaces

$\mathrm{VC}^*(\mathcal{S}) \geq 1$:

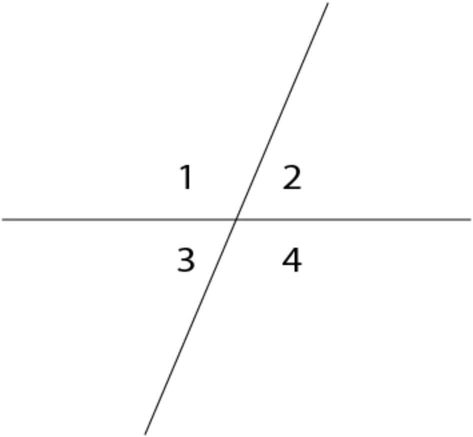


$\mathrm{VC}^*(\mathcal{S}) < 2$:



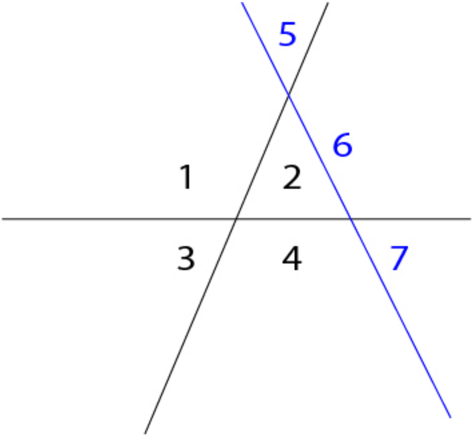
Example: $X = \mathbb{R}^2$, $\mathcal{S} =$ Half-Spaces

$\mathrm{VC}^*(\mathcal{S}) \geq 2$:



Example: $X = \mathbb{R}^2$, $\mathcal{S} =$ Half-Spaces

$\mathrm{VC}^*(\mathcal{S}) < 3$:



Set Systems in a Model-Theoretic Context

Consider a sorted language $\mathcal{L}$ with sorts indexed by I.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure with domains $(M_i : i \in I)$.

Definition

Given an $\mathcal{L}$-formula $\phi(x, y)$ where $x = (x_1^{i_1}, ..., x_s^{i_s})$ and $y = (y_1^{j_1}, ..., y_t^{j_t})$, define

$$\mathcal{S}_\phi = \{\phi(X, b) : b \in Y\}$$

where $X = M_{i_1} \times \cdots \times M_{i_s}$ and $Y = M_{j_1} \times \cdots \times M_{j_t}$.

It follows that $(X, \mathcal{S}_\phi)$ is a set system. To ease notation, we let:

π_ϕ denote $\pi_{\mathcal{S}_\phi}$, $VC(\phi)$ denote $VC(\mathcal{S}_\phi)$, and $vc(\phi)$ denote $vc(\mathcal{S}_\phi)$.

Similarly, we use π_ϕ^* for $\pi_{\mathcal{S}_\phi}^*$, $\mathrm{VC}^*(\phi)$ for $\mathrm{VC}^*(\mathcal{S}_\phi)$, and $\mathrm{vc}^*(\phi)$ for $\mathrm{vc}^*(\mathcal{S}_\phi)$.

Duality in a Model-Theoretic Context

Lemma

The dual shatter function of ϕ is the shatter function of ϕ^*. That is $\pi^*_\phi = \pi_{\phi^*}$.

Proof: For all $n < \omega$, we have

$$\begin{aligned}\pi^*_\phi(n) &= \max\{|S(\phi(X,b) : b \in B)| : B \in [Y]^n\} \\ &= \max\{|S_\phi(B)| : B \in [Y]^n\} \\ &= \max\{|\mathcal{S}_{\phi^*} \cap B| : B \in [Y]^n\} \\ &= \pi_{\phi^*}(n).\end{aligned}$$

$\square$

Corollary

$\mathrm{VC}^*(\phi) = \mathrm{VC}(\phi^*)$ *and* $\mathrm{vc}^*(\phi) = \mathrm{vc}(\phi^*)$.

$\mathrm{VC}(\phi) < \omega \iff \mathrm{VC}^*(\phi) < \omega$

Lemma

$\mathrm{VC}(\phi) < 2^{\mathrm{VC}^*(\phi)+1}$.

Proof: Suppose $\mathrm{VC}(\phi) \geq 2^n$, there exists $A \in [X]^{2^n}$ shattered by $\mathcal{S}_\phi$.

Let $\{a_J : J \subseteq n\}$ enumerate A.

For all $i < n$, let $b_i \in Y$ such that $\mathcal{M} \models \phi(a_J, b_i) \iff i \in J$.

Let $B = \{b_i : i < n\}$.

It follows that $\mathcal{S}_{\phi^*}$ shatters B, so $\mathrm{VC}(\phi^*) \geq n$. □

Corollary

$\mathrm{VC}^*(\phi) < 2^{\mathrm{VC}(\phi)+1}$.

Corollary

$\mathrm{VC}(\phi) < \omega \iff \mathrm{VC}^*(\phi) < \omega$.

Key Point from Last Week

In the classical context...

$\pi^*_{\mathcal{S}}(n)$ is counting atoms generated by n sets in $\mathcal{S}$.

In the model-theoretic context...

$\pi^*_{\phi}(n)$ is counting ϕ-types over n parameters.

So by duality...

$\pi_{\phi}(n)$ is counting ϕ^*-types over n parameters.

Duality in the Classical Context

Given $(X, \mathcal{S})$ a set system, let $\mathcal{M} = (X, \mathcal{S}, \in)$, and $\phi(x, y)$ be $x \in y$.

It follows that $\mathcal{S} = \mathcal{S}_\phi$, so by definition, $\pi_\mathcal{S} = \pi_\phi$ and $\pi^*_\mathcal{S} = \pi^*_\phi$.

Let $X^* = \mathcal{S}$ and

$$\begin{aligned} \mathcal{S}^* &= \{\{B \in \mathcal{S} : a \in B\} : a \in X\} \\ &= \{\phi^*(\mathcal{S}, a) : a \in X\}. \end{aligned}$$

It follows that $\mathcal{S}^* = \mathcal{S}_{\phi^*}$, so by definition, $\pi_{\mathcal{S}^*} = \pi_{\phi^*}$ and $\pi^*_{\mathcal{S}^*} = \pi^*_{\phi^*}$.

Definition

We call $(X^*, \mathcal{S}^*)$ the *dual* of $(X, \mathcal{S})$.

Lemma

$\pi^*_\mathcal{S} = \pi_{\mathcal{S}^*}$ and $\pi^*_{\mathcal{S}^*} = \pi_\mathcal{S}$.

Proof: $\pi^*_\mathcal{S} = \pi^*_\phi = \pi_{\phi^*} = \pi_{\mathcal{S}^*}$ and $\pi^*_{\mathcal{S}^*} = \pi^*_{\phi^*} = \pi_\phi = \pi_\mathcal{S}$.

Duality in the Classical Context

Corollary

$\mathsf{VC}^*(\mathcal{S}) = \mathsf{VC}(\mathcal{S}^*)$ *and* $\mathsf{vc}^*(\mathcal{S}) = \mathsf{vc}(\mathcal{S}^*)$.

Corollary

For any set system $(X, \mathcal{S})$, *we have*

$$\mathsf{VC}(\mathcal{S}) < 2^{\mathsf{VC}^*(\mathcal{S})+1}$$

and

$$\mathsf{VC}^*(\mathcal{S}) < 2^{\mathsf{VC}(\mathcal{S})+1}.$$

Corollary

$\mathsf{VC}(\mathcal{S}) < \omega \iff \mathsf{VC}^*(\mathcal{S}) < \omega$.

Elementary Properties

Lemma

π^*_ϕ is elementary (i.e., elementarily equivalent $\mathcal{L}$-structures agree on π^*_ϕ).

Proof: Given $n < \omega$, let $\sigma \in {}^{\mathcal{P}(n)}2$. Consider the $\mathcal{L}$-sentence

$$\exists y_1, \ldots, y_n \bigwedge_{J \subseteq n} \left[\exists x \bigwedge_{i=1}^{n} \phi^{[i \in J]}(x, y_i) \right]^{\sigma(J)} .$$

Corollary

$\mathrm{VC}^*(\phi)$ *and* $\mathrm{vc}^*(\phi)$ *are elementary.*

Corollary

$\mathrm{VC}(\phi)$ *and* $\mathrm{vc}(\phi)$ *are elementary.*

NIP Formulae

Let T be a complete $\mathcal{L}$-theory, and let $\phi(x, y) \in \mathcal{L}$.

Definition

We say ϕ has the *independence property (IP)* iff: for some $\mathcal{M} \models T$, there exists sequences $(a_i : i < \omega) \subseteq M^{|x|}$ and $(b_J : J \subseteq \omega) \subseteq M^{|y|}$ such that

$$\mathcal{M} \models \phi(a_i, b_J) \quad \iff \quad i \in J.$$

If ϕ is not IP, we say ϕ is *NIP*.

Lemma

ϕ *is IP* $\quad \iff \quad \mathrm{VC}(\phi) = \omega.$

Proof: Compactness.

Corollary

ϕ *is NIP* $\quad \iff \quad \mathrm{VC}(\phi) < \omega \quad \iff \quad \mathrm{VC}^*(\phi) < \omega.$

NIP and vc^T

Let T be a complete $\mathcal{L}$-theory.

Definition

We say T is *NIP* iff: every partitioned $\mathcal{L}$-formula is NIP.

Fact: It is sufficient to check all $\phi(x, y)$ with $|y| = 1$ (and $|x| \geq 1$).

Open Question: Is it possible for $\mathrm{vc}(\phi)$ to be irrational in an NIP theory?

Definition

The *VC density of* T is the function

$$\mathrm{vc}^T : \omega \longrightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$$

defined by

$$\begin{aligned} \mathrm{vc}^T(n) &= \sup\{\mathrm{vc}(\phi) : \phi(x, y) \in \mathcal{L},\ |y| = n\} \\ &= \sup\{\mathrm{vc}^*(\phi) : \phi(x, y) \in \mathcal{L}, |x| = n\}. \end{aligned}$$

NIP and vc^T

Lemma

If $\mathrm{vc}^T(n) < \infty$ for all $n < \omega$, then T is NIP.

Note: Converse is not true in general; e.g., consider T^{eq} where T is NIP.

Open Questions:

1. For every language $\mathcal{L}$ and every complete $\mathcal{L}$-theory T, does $\mathrm{vc}^T(1) < \infty$ imply $\mathrm{vc}^T(n) < \infty$ for all $n < \omega$?
2. If so, is there some bounding function β, independent of $\mathcal{L}$ and T, such that $\mathrm{vc}^T(n) < \beta(\mathrm{vc}^T(1), n)$?

Finite Types

Let $\Delta(x, y)$ be a finite set of $\mathcal{L}$-formulae (with free variables x and y).

Definition

The set system generated by Δ is

$$\mathcal{S}_\Delta = \left\{ \phi\left(M^{|x|}, b\right) : \phi(x,y) \in \Delta,\ b \in M^{|y|} \right\}.$$

The dual shatter function of Δ is

$$\pi^*_\Delta(n) = \max\left\{ |S_\Delta(B)| : B \in \left[M^{|y|}\right]^n \right\}.$$

The dual VC density of Δ is

$$vc^*_\Delta(n) = \inf\{r \in \mathbb{R}^{>0} : \pi^*_\Delta(n) = O(n^r)\}.$$

Fact: π^*_Δ and vc^*_Δ are elementary.

Defining Schemata

Let $\Delta(x, y) \subseteq \mathcal{L}$ and $B \subseteq M^{|y|}$ both be finite. Let $p \in S_\Delta(B)$.

Definition

Given a *schema*

$$d(y, z) = \{d_\phi(y, z) : \phi \in \Delta\} \subseteq \mathcal{L}$$

and a parameter $c \in M^{|z|}$, we say that $d(y, c)$ *defines* p iff:
for every $\phi \in \Delta$ and $b \in B$, we have

$$\phi(x, b) \in p \quad \Longleftrightarrow \quad \mathcal{M} \models d_\phi(b, c).$$

UDTFS and the VC n Property

Let $\Delta(x, y) \subseteq \mathcal{L}$ be finite.

Definition

We say Δ has *uniform definability of types over finite sets (UDTFS) with n parameters* iff: there is a finite family $\mathcal{F}$ of schemata each of the form

$$d(y, z_1, ..., z_n) = \{d_\phi(y, z_1, ..., z_n) : \phi \in \Delta\}$$

with $|y| = |z_1| = \cdots = |z_n|$ such that if $B \subseteq M^{|y|}$ is finite and $p(x) \in S_\Delta(B)$, then for some $d \in \mathcal{F}$ and $b_1, ..., b_n \in B$, $d(y, \overline{b})$ defines p.

Definition

An $\mathcal{L}$-structure has the *VC n property* iff:
all finite $\Delta(x, y) \subseteq \mathcal{L}$ with $|x| = 1$ have UDTFS with n parameters.

Fact: UDTFS with n parameters and VC n are elementary properties.

Uniform Bounds on VC Density

Theorem (5.7)

If $\mathcal{M}$ has the VC n property, then every finite $\Delta(x, y) \subseteq \mathcal{L}$ has UDTFS with $n|x|$ parameters.

Corollary (5.8a)

If $\mathcal{M}$ has the VC n property, then for every finite $\Delta(x, y) \subseteq \mathcal{L}$, we have $\mathrm{vc}^*(\Delta) \leq n|x|$.

Proof: Given $\Delta(x, y)$ finite, there exists finite $\mathcal{F}$ witnessing UDTFS with $n|x|$ parameters. It follows that $|S_\Delta(B)| \leq |\mathcal{F}||B|^{n|x|}$.

Corollary (5.8b)

If T is complete and has the VC n property, then for all $m < \omega$, we have $\mathrm{vc}^T(m) \leq nm$.

Weakly O-Minimal Theories are VC 1

Theorem (6.1)

If T is complete and weakly o-minimal, then T has the VC 1 property.

Proof: Let $\mathcal{M} \models T$, and let $\Delta(x, y) \subseteq \mathcal{L}$ be finite with $|x| = 1$.

By Compactness, there exists $n < \omega$ such that for all $\phi \in \Delta$ and $b \in M^{|y|}$,

$$\phi(M, b) \text{ has at most } n \text{ maximal convex components.}$$

For all $\phi \in \Delta$ and $i < n$, there exists $\phi_i(x, y) \in \mathcal{L}$ such that for each $b \in M^{|y|}$,

$$\phi_i(M, b) \text{ is the } i^{th} \text{component of } \phi(M, b).$$

It follows that

$$\mathcal{M} \models \phi(x, y) \ \leftrightarrow \ \bigvee_{i<n} \phi_i(x, y).$$

Proof of Theorem (cont.)

For each $\phi \in \Delta$ and $i < n$, let

$$\begin{array}{lll} \phi_i^{\leq}(x,y) & \text{be} & \exists x_0\,[\phi_i(x_0,y) \;\wedge\; x \leq x_0] \\ \phi_i^{<}(x,y) & \text{be} & \forall x_0\,[\phi_i(x_0,y) \;\rightarrow\; x < x_0]. \end{array}$$

It follows that

$$\mathcal{M} \models \phi_i(x,y) \quad \leftrightarrow \quad \phi_i^{\leq}(x,y) \;\wedge\; \neg\phi_i^{<}(x,y).$$

If we let $\Psi = \{\phi_i^{<},\ \phi_i^{\leq} \ :\ \phi \in \Delta,\ i < n\}$, each formula in Δ is T-equivalent to a boolean combination of $2n$ formulae in Ψ.

For each $\psi \in \Psi$ and $b \in M^{|y|}$, notice that $\psi(M,b)$ is an initial segment of M, so $\mathcal{S}_\Psi$ is directed.

Lemma 5.2 $\Rightarrow$ Ψ has UDTFS with one parameter.

Lemma 5.5 $\Rightarrow$ Δ has UDTFS with one parameter.

Application: RCVF

Let $\mathcal{L} = \{+\,,-\,,\cdot\,,0\,,1\,,<\,,|\}$.

RCVF (with a proper convex valuation ring) where $|$ is the divisibility predicate (i.e., $a|b \;\Leftrightarrow\; v(a) \leq v(b)$) is a complete $\mathcal{L}$-theory.

Cherlin and Dickmann showed RCVF has quantifier elimination and is, therefore, weakly *o*-minimal.

Corollary (6.2a)

In RCVF, *if* $\Delta(x, y) \subseteq \mathcal{L}$ *is finite, then* $\mathrm{vc}^*(\Delta) \leq |x|$.

Corollary (6.2b)

$\mathrm{vc}^{\mathrm{RCVF}}(n) \leq n$ *for all* $n < \omega$.

Application: $\mathrm{ACVF}_{(0,0)}$

Let $\mathcal{L} = \{+\,,-\,,\cdot\,,0\,,1\,,|\}$.

$\mathrm{ACVF}_{(0,0)}$ where $|$ is the divisibility predicate is complete in $\mathcal{L}$.

Let $R \models \mathrm{RCVF}$ (in $\mathcal{L} \cup \{<\}$).

Consider $R(i)$ where $i^2 = -1$ and

$$a + bi \mid c + di \;\Leftrightarrow\; a^2 + b^2 \mid c^2 + d^2.$$

It follows that $R(i) \models \mathrm{ACVF}_{(0,0)}$ and is interpretable in R.

Corollary (6.3a)

In $\mathrm{ACVF}_{(0,0)}$, *if* $\Delta(x, y) \subseteq \mathcal{L}$ *is finite, then* $\mathrm{vc}^*(\Delta) \leq 2|x|$.

Corollary (6.3b)

$\mathrm{vc}^{\mathrm{ACVF}_{(0,0)}}(n) \leq 2n$, *for all* $n < \omega$.

Open Questions

1 For every language $\mathcal{L}$ and every complete $\mathcal{L}$-theory T, does $\mathrm{vc}^T(1) < \infty$ imply $\mathrm{vc}^T(n) < \infty$ for all $n < \omega$?

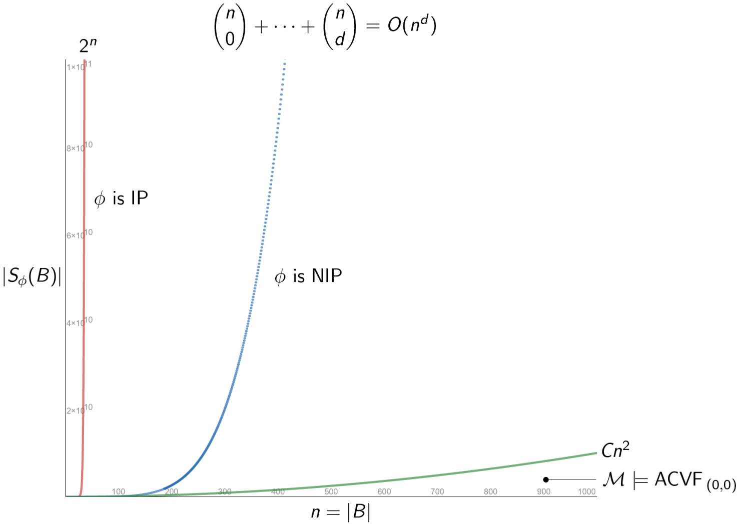
$$\begin{array}{rcl} \mathrm{RCVF} & : & \text{Yes} \\ \mathrm{ACVF}_{(0,0)} & : & \text{Yes} \end{array} \qquad \begin{array}{rcl} \mathrm{ACVF}_{(0,p)} & : & ? \\ \mathrm{ACVF}_{(p,p)} & : & ? \end{array}$$

2 If so, is there some bounding function β, independent of $\mathcal{L}$ and T, such that $\mathrm{vc}^T(n) < \beta(\mathrm{vc}^T(1), n)$?

$$\begin{array}{rcl} \mathrm{RCVF} & : & \beta(n) = n \\ \mathrm{ACVF}_{(0,0)} & : & \beta(n) = 2n \end{array} \qquad \begin{array}{rcl} \mathrm{ACVF}_{(0,p)} & : & ? \\ \mathrm{ACVF}_{(p,p)} & : & ? \end{array}$$

Counting Types

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, $\phi(x,y) \in \mathcal{L}$ with $|x| = 1$, and $B \subseteq M^{|y|}$.

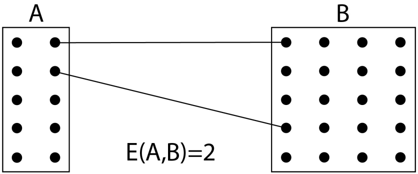


Regular Pairs

Let $G = (V, E)$ be a finite graph. Fix $\varepsilon, \delta \in [0, 1]$.

Definition

Given $A, B \subseteq V$, we say the pair (A, B) is (ε, δ)-*regular* iff: there exists $\alpha \in [0, 1]$ such that for all nonempty sets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \delta|A|$ and $|B'| \geq \delta|B|$, we have $|d(A', B') - \alpha| \leq \frac{\varepsilon}{2}$.



$$d(A, B) = \frac{E(A,B)}{|A||B|} = \frac{2}{200} = 0.01$$

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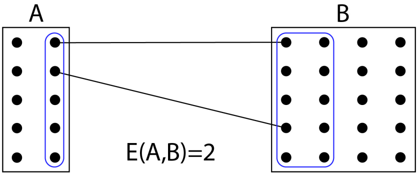

$$d(A, B) = \frac{E(A,B)}{|A||B|} = \frac{2}{200} = 0.01$$

$$\alpha = 0 \qquad \delta = 0.5 \qquad \epsilon = 0.04$$

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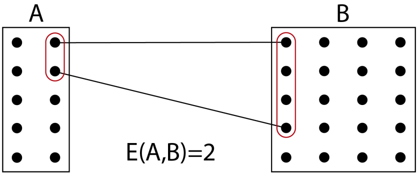


$$d(A, B) = \frac{E(A,B)}{|A||B|} = \frac{2}{200} = 0.01$$

$\alpha = 0$ $\qquad \delta = 0.2$ $\qquad \epsilon = 0.25$

Defect

Let P be a finite partition of V. Fix $\eta \in [0,1]$.

Definition

The *defect* of P is

$$\mathrm{def}_{\varepsilon,\delta}(P) := \{(A,B) \in P^2 : (A,B) \text{ not } (\varepsilon,\delta)\text{-regular}\},$$

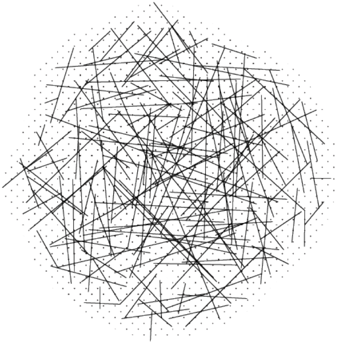
and we say that P is $(\varepsilon,\delta,\eta)$-regular iff:

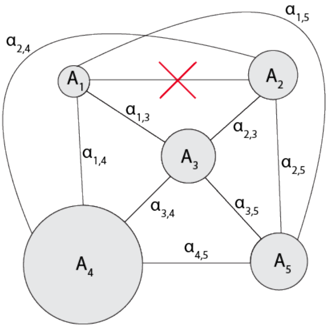
$$\sum_{(A,B)\,\in\,\mathrm{def}_{\varepsilon,\delta}(P)} |A||B| \leq \eta|V|^2.$$

Szemerédi Regularity Lemma (without Equipartition)

Lemma

For all $\varepsilon, \delta, \eta > 0$, there exists $M = M(\varepsilon, \delta, \eta)$ such that any finite graph has an $(\varepsilon, \delta, \eta)$-regular partition with at most M parts.





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(Szemerédi 1976) $M(\varepsilon, \varepsilon, \varepsilon) \leq \mathrm{twr}_2(O(\varepsilon^{-5}))$

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How fast does M grow as $\delta \to 0$?

(Gowers 1997) $M(1 - \delta^{1/16}, \delta, 1 - 20\delta^{1/16}) \geq \mathrm{twr}_2(\Omega(\delta^{-1/16}))$

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Definable Regularity Lemma for NIP Relations

Let $k \geq 2$ and $d \in \mathbb{N}$.

Theorem

There is a constant $c = c(k, d)$ such that IF

- $\varepsilon, \delta, \eta > 0$
- $E = \phi(\overline{V})$ *for some* $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$ *and structure* $\mathcal{M}$
- $\mathrm{VC}(E) \leq d$
- *each μ_i is a Keisler measure on V_i which is fap on E*

THEN there is an $(\varepsilon, \delta, \eta)$-regular partition $\overline{P}$ of $\overline{V}$ with 0-1 densities such that

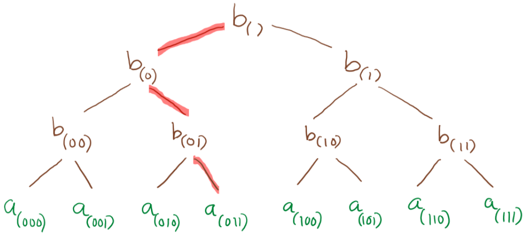
- $\|\overline{P}\| \leq O(\gamma^{-c})$ *where* $\gamma = \min\{\varepsilon, \delta, \eta\}$
- *for each P_i, the parts of P_i are definable using a single formula ψ_i which is a boolean combination of ϕ depending only on γ and ϕ.*

Stability

Let $d \in \mathbb{N}$ and $R \subseteq V \times W$.

Definition

We say R is d-stable iff: there is not a tree of parameters $\{b_\tau : \tau \in {}^{<d}2\} \subseteq W$ along with a set of leaves $\{a_\sigma : \sigma \in {}^{d}2\} \subseteq V$ such that for any $\sigma \in {}^{d}2$ and $n < d$, we have $(a_\sigma, b_{\sigma\restriction_n}) \in R \iff \sigma(n) = 1$.



$$a_{(011)} \models \neg R(x, b_{()}) \wedge R(x, b_{(0)}) \wedge R(x, b_{(01)})$$

Definable Regularity Lemma for Stable Relations

Let $k \geq 2$ and $d \in \mathbb{N}$.

Theorem

There is a constant $c = c(k, d)$ such that IF

- $\varepsilon, \delta > 0$ *and* $\eta = 0$
- $E = \phi(\overline{V})$ *for some* $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$ *and structure* $\mathcal{M}$
- E *is* d*-stable*
- *each* μ_i *is a Keisler measure on* V_i *which is fap on* E

THEN there is an $(\varepsilon, \delta, \eta)$*-regular partition* $\overline{P}$ *of* $\overline{V}$ *with* 0-1 *densities such that*

- $\|\overline{P}\| \leq O(\gamma^{-c})$ *where* $\gamma = \min\{\varepsilon, \delta\}$
- *for each* P_i*, the parts of* P_i *are definable using a single formula* ψ_i *which is a boolean combination of* ϕ *depending only on* γ *and* ϕ*.*

Distality

Let T be a complete NIP theory and $\mathcal{U}$ a monster model for T.

Definition

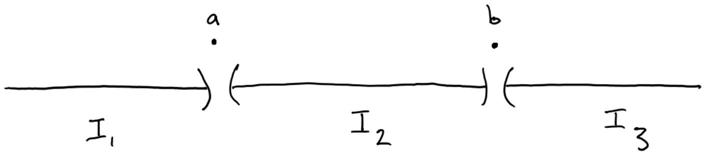
We say T is distal iff: for all $n \geq 1$, all indiscernible sequences $I \subseteq U^n$, and all Dedekind cuts $I = I_1 + I_2 + I_3$, if

$$I_1 + a + I_2 + I_3 \quad \text{and} \quad I_1 + I_2 + b + I_3$$

are both indiscernible, then

$$I_1 + a + I_2 + b + I_3$$

is also indiscernible.



Definable Regularity Lemma for Distal NIP Structures

Let T be a complete distal NIP theory and $\mathcal{M} \models T$.

Let $k \geq 2$ and $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$.

Theorem

There is a constant $c = c(\mathcal{M}, \phi)$ such that IF

- $\varepsilon = \delta = 0$ *and* $\eta > 0$
- $E = \phi(\overline{V})$
- *each μ_i is a Keisler measure on V_i which is fap on E*

THEN there is an $(\varepsilon, \delta, \eta)$-regular partition $\overline{P}$ of $\overline{V}$ with 0-1 densities such that

- $\|\overline{P}\| \leq O(\eta^{-c})$
- *for each P_i, the parts of P_i are definable using a single formula ψ_i which is a boolean combination of ϕ depending only on ϕ.*